Excel Journal of Engineering Technology and Management Science
(An International Multidisciplinary Journal)
Vol. I No.6 January-June 2014 (Online) ISSN 2277-3339

NUMERICAL SOLUTION OF FUZZY FREDHOLM INTEGRAL EQUATIONS OF SECOND KIND

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Introduction

The concept of integration of fuzzy functions was first introduced by Dubois and Prade [5]. The topics of numerical methods for solving fuzzy integral equations have been rapidly growing in recent years and have been studied by M. Friedman, M. Ma, A. Kandel [6]. The numerical methods for fuzzy differential equations have been studied by S. Abbasbandy, T. Allahviranloo, [1, 2, 3] and others. Alternative approaches were later suggested by Goetschel and Vaxman [8], Kaleva [10] and others. The structure of this paper is organized as follows:

In section 2, some basic definitions and results on fuzzy numbers, fuzzy integral and the fuzzy linear system is brought. In Section 3, we propose a general method for solving fuzzy Fredholm integral equation of the second kind. In Section 4, we propose a residual minimization method for solving fuzzy Fredholm integral equation of the second kind.In Section 5, we illustrate algorithm by solving some numerical examples. The conclusions are drawn in Section 6.

Preliminaries

Let us now introduce the notation needed in the rest of the paper. We will place a wavy bar over a symbol if it represents a fuzzy number so $\tilde{a}, \tilde{b}, \tilde{c}$ are all fuzzy numbers but *a, b, c* will denote real numbers. Parametric form of an arbitrary fuzzy number is given in [4] as follows. A fuzzy number $\tilde{u}$ in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$, $\overline{u}(r)$, $0 \le r \le 1$, which satisfies the following requirements:

1. $\underline{u}(r)$ is a bounded left continuous non-decreasing function over [0, 1],
2. $\overline{u}(r)$ is a bounded left continuous non-increasing function over [0, 1],
3. $\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$,

The set of all these fuzzy numbers is denoted by E. A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha$, $0 \le r \le 1$.

A popular fuzzy number is the triangular fuzzy number $\tilde{u} = (m, \alpha, \beta)$ which

$$\tilde{u}(x) = \begin{cases} \dfrac{x-m}{\alpha} + 1, & m - \alpha \le x \le m, \\ \dfrac{x-m}{\beta} + 1, & m \le x \le m + \beta, \\ 0, & Otherwise \end{cases}$$

Its parametric form is

$$\underline{u}(r) = m + \alpha(r - 1),\ \overline{u}(r) = m + \beta(1 - r).$$

By appropriate definitions the fuzzy number space $\{\underline{u}(r),\ \overline{u}(r)\}$ becomes a convex cone E^1 which is then embedded isomorphically and isometrically into a Banach space.

Definition 1. The $n \times n$ dual linear system

$$\left.\begin{array}{l} a_{11}\tilde{x}_1 + \ldots + a_{1n}\tilde{x}_n = \tilde{y}_1 + b_{11}\tilde{x}_1 + \ldots + b_{1n}\tilde{x}_n \\ a_{21}\tilde{x}_1 + \ldots + a_{2n}\tilde{x}_n = \tilde{y}_2 + b_{21}\tilde{x}_1 + \ldots + b_{2n}\tilde{x}_n \\ \vdots \qquad\qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{n1}\tilde{x}_1 + \ldots + a_{nn}\tilde{x}_n = \tilde{y}_n + b_{n1}\tilde{x}_1 + \ldots + b_{nn}\tilde{x}_n \end{array}\right\}$$

where the coefficient matrix $A = (a_{ij})$ and $B = (b_{ij})$, $1 \le i, j \le n$ is a crisp $n \times n$ matrix, $\tilde{x}^t = (\tilde{x}_1, \ldots, \tilde{x}_n)$ be a $n \times 1$ vector of fuzzy numbers $\tilde{x}_j$ and $\tilde{y}^t = (\tilde{y}_1, \ldots, \tilde{y}_n)$ be a $n \times 1$ vector of fuzzy numbers $\tilde{y}_i$ is called a dual fuzzy linear system (DFLS).

For arbitrary fuzzy numbers $\tilde{x} = (\underline{x}(r), \overline{x}(r))$, $\tilde{y} = (\underline{y}(r), \overline{y}(r))$ and real number k, we may define the addition and the scalar multiplication of fuzzy numbers by using the extension principle as

(a) $\tilde{x} = \tilde{y}$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$,

(b) $\tilde{x} + \tilde{y} = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$,

(c) $k\tilde{x} = \begin{cases} (k\underline{x}, k\overline{x}), & k \ge 0 \\ (k\overline{x}, k\underline{x}), & k < 0 \end{cases}$

Definition 2: A fuzzy number vector $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)^t$ given by

$\tilde{x}_i = (\underline{x}_i(r), \overline{x}_i(r)),\ 1 \le i \le n,\ 0 \le r \le 1,$

is called a solution of the fuzzy linear system (1) if

If, for a particular i, $a_{ij} > 0$ and $b_{ij} > 0$, $1 \le j \le n$, we simply get

$$\sum_{j=1}^{n} a_{ij}\underline{x}_j = \underline{y}_i + \sum_{j=1}^{n} b_{ij}\underline{x}_j,\quad \sum_{j=1}^{n} a_{ij}\overline{x}_j = \overline{y}_i + \sum_{j=1}^{n} b_{ij}\overline{x}_j.$$

Consider the dual fuzzy linear system and transform its $n \times n$ coefficient matrix A and B in to $(2n) \times (2n)$ matrices as:

$$SX = Y + TX,$$

therefore, we have:

$$(S - T)X = Y,$$

where $S = (s_{ij}) \ge 0$ and $T = (t_{ij}) \ge 0$, $1 \le i, j \le 2n$, and

$$X = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ -\overline{x}_1 \\ \vdots \\ -\overline{x}_n \end{bmatrix}, \qquad Y = \begin{bmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_n \\ -\overline{y}_1 \\ \vdots \\ -\overline{y}_n \end{bmatrix}.$$

Let $(S - T)X = Y$

Where structure of S and T is as:

$$S = \begin{pmatrix} C & D \\ D & C \end{pmatrix}, \qquad T = \begin{pmatrix} E & F \\ F & E \end{pmatrix}$$

and C and E contains the positive entries of A and B respectively, and D and F the absolute values of the negative entries of A and B, i.e. $A = C - D$ and $B = E - F$. Therefore

$$S - T = \begin{pmatrix} C-E & D-F \\ D-F & C-E \end{pmatrix},$$

The matrix $S - T$ is nonsingular if and if the matrix $(C + D) - (E + F)$ and $(C + F) - (E + D)$ are both nonsingular.

$$X = (S - T)^{-1}Y \qquad \ldots (1)$$

If $(S - T)^{-1}$ exists it must have the same structure as S, i.e.

$$(S - T)^{-1} = \begin{pmatrix} G & H \\ H & G \end{pmatrix},$$

and

$$G = \frac{1}{2}\left[\left((C+D)-(E+F)\right)^{-1} + \left((C+F)-(E+D)\right)^{-1}\right],$$

$$H = \frac{1}{2}\left[\left((C+D)-(E+F)\right)^{-1} - \left((C+F)-(E+D)\right)^{-1}\right]$$

The unique solution X of equation (1) is a fuzzy vector for arbitrary Y if and only if $(S - T)^{-1}$ is nonnegative, i.e.

$$((S - T)^{-1})_{ij} \geq 0,\ 1 \leq i \leq 2n,\ 1 \leq j \leq 2n.$$

Fuzzy integral equations of second kind

The Fredholm integral equation of the second kind is

$$x(s) = f(s) + \lambda \int_a^b k(s,t)\, x(t)\, dt \qquad \ldots (2)$$

where $\lambda > 0$, $k(s, t)$ is an arbitrary kernel function over the square $a \leq s, t \leq b$ and $f(t)$ is a function of $t : a \leq t \leq b$, [9]. If $f(t)$ is a crisp function then the solution of equation (2) is crisp as well. However, if $f(t)$ is a fuzzy function this equation may only possess fuzzy solution. Therefore, we have

$$\tilde{x}(s) = \tilde{f}(s) + \lambda \int_a^b k(s,t)\,\tilde{x}(t)dt \qquad \ldots (3)$$

Sufficient conditions for the existence of a unique solution to the fuzzy Fredholm integral equation of the second kind, i.e. to equation (3) where $\tilde{f}(t)$ is a fuzzy function, are given in [13].

We consider now the numerical solution of fuzzy Fredholm integral equations of the second kind equation (3), which we write in the form:

$$\tilde{x} = \tilde{f} + \lambda K \tilde{x}$$

The exact solution of integral equation, equation (3) is:

$$\tilde{x}(s) = \sum_{i=1}^{\infty} \tilde{a}_i h_i(s)$$

in truncated form

$$\tilde{x}(s) \approx \tilde{x}_n(s) = \sum_{i=1}^{n} \tilde{a}_i h_i(s) \qquad \ldots (4)$$

where the set $\{h_i\}$ is complete and orthogonal in $\ell^2(a,b)$. For finding approximation solution we must indicate coefficients $\tilde{a}_i$.

From equation (4) we obtain

$$\sum_{j=1}^{n} \tilde{a}_j h_j(s) = \tilde{f}(s) + \lambda \sum_{j=1}^{n} \tilde{a}_j \int_a^b k(s,t) h_j(t)dt.$$

We have n unknown parameters in the form $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n$ which for finding them, we need to n equation, so by using n point $s_1, s_2, \ldots, s_n$ in interval $[a, b]$:

$$\sum_{j=1}^{n} h_j(s_i)\tilde{a}_j = \tilde{f}(s_i) + \lambda \sum_{j=1}^{n} \tilde{a}_j \int_a^b k(s_i,t) h_j(t)dt\tilde{a}_j \qquad i = 1,\ldots,n \qquad \ldots (5)$$

therefore we have:

$$A\tilde{a} = \tilde{f} + B\tilde{a}\,, \qquad \ldots (6)$$

where the coefficients matrix $A = (a_{ij})$, $1 \le i, j \le n$, and $B = (b_{ij})$, $1 \le i, j \le n$, are crisp and $\tilde{f} = (\tilde{f}_i)$, $1 \le i \le n$, is an arbitrary fuzzy number vector, where

$$a_{ij} = h_j(s_i)\,, \qquad b_{ij} = \lambda \int_a^b k(s_i,t) h_j(t)dt\,, \qquad i, j = 1,\ldots,n$$

Residual minimization method

The simplest method conceptually again appeals to approximation theory. We write the integral equation in the form (again we set $\lambda = 1$)

$$L\tilde{x} = \tilde{f}\,, \qquad L = I - K, \qquad \ldots (7)$$

and introduce the residual function r_n and error function ε_n

$$r_n = \dot{D}(\tilde{f}, L\tilde{x}_n), \qquad \ldots (8)$$

$$\varepsilon_n = \dot{D}(\tilde{x}, \tilde{x}_n)$$

To compute r_n requires no knowledge of $\tilde{x}$ but, since $\dot{D}(\tilde{f}, L\tilde{x}) = 0$, we have the identity:

$$r_n = \dot{D}(\tilde{f}, L\tilde{x}_n) - \dot{D}(\tilde{f}, L\tilde{x}) = L\dot{D}(\tilde{x}_n, \tilde{x}) = L\varepsilon_n \qquad \ldots (9)$$

From equations (7) and (8) we have at once

$$\|r_n\| \le (1 + \|K\|)\|\varepsilon_n\|.$$

That is,

$$\|\varepsilon_n\| \ge \frac{\|r_n\|}{1 + \|K\|}.$$

Thus a small residual is a necessary condition for a small error. We would rather have an upper bound on ε_n of course; this is harder to provide in general and we content ourselves for now with the following. We rewrite (9) as

$$\varepsilon_n = r_n + K\varepsilon_n$$

Hence

$$\|\varepsilon_n\| \le \|r_n\| + \|K\|.\|\varepsilon_n\| \qquad \text{if} \qquad \|K\| < 1$$

$$\|\varepsilon_n\| \le \frac{\|r_n\|}{1 - \|K\|}.$$

Illustration: Let the fuzzy integral equation (3) where

$$\underline{f}(s;r) = \left(\frac{s}{2} - \frac{1}{3}\right)r, \qquad \bar{f}(s;r) = \left(\frac{s}{2} - \frac{1}{3}\right)(2 - r),$$

and kernel

$$k(s,t) = s + t,\ 0 \le s,\ t \le 1.$$

and $a = 0$, $b = 1$. The exact solution in this case is

$$\underline{x}(s;r) = s\,r,$$

$$\underline{x}(s;r) = s(2 - r).$$

Let

$$h_1(s) = s,\ h_2(s) = s^3,\ h_3(s) = s^5,$$

and

$$s_1 = 0,\ s_2 = \frac{1}{2},\ s_3 = 1.$$

From the equations (5) and (6):

$$\begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{8} & \frac{1}{32} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \end{bmatrix} = \begin{bmatrix} \tilde{f}(s_1) \\ \tilde{f}(s_2) \\ \tilde{f}(s_3) \end{bmatrix} + \begin{bmatrix} \frac{1}{3} & \frac{1}{5} & \frac{1}{7} \\ \frac{7}{12} & \frac{13}{40} & \frac{19}{84} \\ \frac{5}{6} & \frac{9}{20} & \frac{13}{42} \end{bmatrix} \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \end{bmatrix},$$

The extended 6×6 matrices are

$$S=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{8} & \frac{1}{32} & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{8} & \frac{1}{32} \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad T=\begin{bmatrix} \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & 0 & 0 & 0 \\ \frac{7}{12} & \frac{13}{40} & \frac{19}{84} & 0 & 0 & 0 \\ \frac{5}{6} & \frac{9}{20} & \frac{13}{42} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{5} & \frac{1}{7} \\ 0 & 0 & 0 & \frac{7}{12} & \frac{13}{40} & \frac{19}{84} \\ 0 & 0 & 0 & \frac{5}{6} & \frac{9}{20} & \frac{13}{42} \end{bmatrix}$$

And the solution of equation (7) is:

$$\begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \underline{a}_3 \\ -\overline{a}_1 \\ -\overline{a}_2 \\ -\overline{a}_3 \end{bmatrix} = (S-T)^{-1}F = \begin{bmatrix} r \\ 0 \\ 0 \\ -2+r \\ 0 \\ 0 \end{bmatrix}, \qquad \ldots (10)$$

That is

$$\tilde{a}_1=(r,2-r), \qquad \tilde{a}_2=(0,0), \qquad \tilde{a}_3=(0,0).$$

Here $\underline{a}_1 \le \overline{a}_1$, $\underline{a}_2 \le \overline{a}_2$, $\underline{a}_3 \le \overline{a}_3$ are monotonic decreasing functions. Therefore the fuzzy solution of the equation (10) is a strong fuzzy solution. The fuzzy approximate solution in this case is a approximate solution is

$$\tilde{x}(s) \approx \tilde{x}_3(s) = \sum_{i=1}^{3} \tilde{a}_i \, h_i(s).$$

Conclusions

In this paper, we proposed a numerical method for solving fuzzy Fredholm integral equation of the second kind. The resulting approximate solutions from expansion method may be strong or weak fuzzy solutions.

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